Solution 5

1. Show that whenever d is a metric defined on X, then

$$\rho(x,y) \equiv \frac{d(x,y)}{1+d(x,y)}$$

is also a metric on X. A sequence converges in d if and only if it converges in ρ.

Solution. M1 and M2 are obvious since d is a metric. To prove M3 consider the function $\phi(x) = x/(1+x)$. We need to show that $a \le b+c$ implies $\phi(a) \le \phi(b)+\phi(c)$. First observe that ϕ is increasing so $\phi(a) \le \phi(b+c)$ when $a \le b+c$. Then

$$\begin{aligned} \phi(b+c) &= \frac{b+c}{1+b+c} \\ &= \frac{b}{1+b+c} + \frac{c}{1+b+c} \\ &\le \frac{b}{1+b} + \frac{c}{1+c} \\ &= \phi(b)+\phi(c)\ , \end{aligned}$$

done.

Next, let $x_n \to x$ in d, we claim $x_n \to x$ in ρ. In fact, $x_n \to x$ means $d(x_n,x) \to 0$. By Limit Theorem,

$$\lim_{n\to\infty} \rho(x_n,x) = \frac{\lim_{n\to\infty} d(x_n,x)}{1+\lim_{n\to\infty} d(x_n,x)} = \frac{0}{1} = 0\ ,$$

so $x_n \to x$ in ρ. Conversely, using the relation

$$d(x,y) = \frac{\rho(x,y)}{1-\rho(x,y)}\ ,$$

and applying a similar argument.

Note. d and ρ may not be equivalent. Consider the standard d on $\mathbb{R}$ which is given by $d(x,y) = |x-y|$. Observe that $\rho(x,y) < 1$ for all x,y. Although trivially $\rho \le d$, the other inequality $d \le C\rho$ cannot hold for any constant C. For, if then we would have $|x-y| = d(x,y) \le C$ for all x,y!

2. Show that d_2 is stronger than d_1 on $C[a,b]$ but they are not equivalent. Hint: Construct a sequence $\{f_n\}$ in $C[0,1]$ satisfying $\|f_n\|_1 \to 0$ but $\|f_n\|_2 \to \infty$ as $n \to \infty$.

Solution. Letting $f,g \in C[a,b]$, by Cauchy-Schwarz inequality,

$$d_1(f,g) = \int_a^b |f-g| \le \sqrt{\int_a^b 1}\ \sqrt{\int_a^b (f-g)^2} = \sqrt{b-a}\ d_2(f,g),$$

so d_2 is stronger than d_1. Next, define f_n as an even function so that $f_n(x) = 0$ for $x \ge 1$, $f_n(0) = n^{3/4}$ and linear between $[0,1/n]$. Then $\{f_n\}$ satisfies our requirement.

Note. In general, it is true that d_q is strictly stronger than d_p when $p<q$ on $C[a,b]$.

3. Consider the functional Φ defined on $C[a, b]$

$$\Phi(f) = \int_a^b \sqrt{1 + f^2(x)}\, dx.$$

Show that it is continuous in $C[a, b]$ under both the supnorm and the L^1-norm. A real-valued function defined on a space of functions is traditionally called a functional.

Solution. Let $h(y) = \sqrt{1 + y^2}$. Then $\Phi(f) = \int_a^b h(f)dx$. Since $h'(y) = \dfrac{y}{\sqrt{1+y^2}} \le 1$, one has, by the mean value theorem

$$\begin{aligned}|\Phi(f) - \Phi(g)| &\le \int_a^b |h(f) - h(g)|dx \le \int_a^b |f - g| \max_{s\in(g,f)} |h'(s)|dx \\ &\le \int_a^b |f - g|dx.\end{aligned}$$

Hence it is continuous in $C[a, b]$ under both the d_1-distance. As d_∞ is stronger than d_1, the functional is also continuous in d_∞.

4. Consider the functional Ψ defined on $C[a, b]$ given by $\Psi(f) = f(x_0)$ where $x_0 \in [a, b]$ is fixed. Show that it is continuous in the supnorm but not in the L^1-norm. Suggestion: Produce a sequence $\{f_n\}$ with $\|f_n\|_1 \to 0$ but $f_n(x_0) = 1$, $\forall n$. Ψ is called an evaluation map.

Solution. $|\Psi(f) - \Psi(g)| = |f(0) - g(0)| \le \max_{x\in[-1,1]} |f(x) - g(x)|$. Hence it is continuous in the d_∞-metric. Let f_n be continuous function such that $f_n(x) = 1, x \in [-1/n, 1/n]; f_n(x) = 0, x \in [-2/n, 2/n]$, and $0 \le f_n \le 1$. Then $\Psi(f_n) = 1$ but $f_n \to 0$ in the d_1-metric.

5. Let Φ be a continuously differentiable function on $\mathbb{R}$. Define a function from $C[0, 1]$ to itself by $G(f)(x) = \Phi(f(x))$. Show that G is continuous.

Solution. For $f_n \to f \in C[0, 1]$, let $M = \max |f|$ and $L = \max\{|\Phi'(z)| : z \in [-M + 1, M + 1]\}$. By Mean-Value Theorem,

$$|\Phi(f_n(x)) - \Phi(f(x))| = |\Phi'(c)(f_n(x) - f(x))| \le L|f_n(x) - f(x)| \le L\|f_n - f\|_\infty .$$

Taking sup over all x, we get

$$\|\Phi \circ f_n - \Phi \circ f\|_\infty \le L\|f_n - f\|_\infty ,$$

and the conclusion follows. Note that we have used the fact that for large n, $f_n(x) \in [-M - 1, M + 1]$, so that Mean-Value Theorem can be applied.

6. Let K be a continuous function defined on $[0, 1] \times [0, 1]$ and consider the map

$$T(f)(x) = \int_0^1 K(x, y)f(y)dy .$$

Show that this map maps $(C[0, 1], \|\cdot\|_1)$ to $(C[0, 1], \|\cdot\|_\infty)$ continuously.

Solution. Let $f_n \to f$ in L^1-norm. We have

$$|T(f_n(x)) - T(f(x))| = \left|\int K(x, y)(f_n(y) - f(y))\, dy\right| \le M\|f_n - f\|_1 ,$$

where M is the supremum or maximum of K. Taking sup over all x on the left, we get

$$\|T(f_n) - T(f)\|_\infty \le K\|f_n - f\|_1 \ ,$$

so T is continuous as asserted.

Almost forgot to verify that $T \circ f \in C[0,1]$. Using the uniform continuity of K, for $\varepsilon > 0$, there is some δ such that $|K(x,y) - K(x',y')| < \varepsilon$ for $|(x,y) - (x',y')| < \delta$. Therefore,

$$|T(f(x_1)) - T(f(x_2))| \le \int_0^1 |K(x_1,y) - K(x_2,y)|\,|f(y)|\;dy \le C\varepsilon,$$

where

$$C = \int_0^1 |f(y)|\;dy$$

is a constant. So $T \circ f$ is continuous.

7. Let A and B be two sets in (X,d) satisfying $d(A,B) > 0$ where

$$d(A,B) \equiv \inf\{d(x,y) :\ (x,y) \in A \times B\}\,.$$

Show that there exists a continuous function f from X to $[0,1]$ such that $f \equiv 0$ in A and $f \equiv 1$ in B. This problem shows that there are many continuous functions in a metric space.

Solution. Let $d_0 = d(A,B) > 0$. Fix a continuous function φ satisfies $\varphi(0) = 0, \varphi(x) = 1, x \ge d_0$, and $0 \le \varphi \le 1$ on $[0,\infty)$. Our desired function is given by $\varphi(d(x,A))$ after noting that the composition of continuous functions is again continuous.

Note. Taking $A = \{x_1\}$ and $B = \{x_2\}$ be singleton sets consisting distinct points, $d(A,B) > 0$ clearly holds. By this problem there is a continuous function which is 0 at x_1 and 1 at x_2, showing that there are many many continuous real-valued functions on a metric space.

8. In class we showed that the set $P = \{f :\ f(x) > 0, \forall x \in [a,b]\}$ is an open set in $C[a,b]$. Show that it is no longer true if the norm is replaced by the L^1-norm. In other words, for each $f \in P$ and each $\varepsilon > 0$, there is some continuous g which is negative somewhere such that $\|g - f\|_1 < \varepsilon$.

Solution. Fix a point, say, a and consider the continuous piecewise function φ_k which is equal to 1 at a and vanishes on $[a + 1/k, b]$. Then

$$\int_a^b \varphi_k(x)dx = \frac{1}{2k}\ .$$

Let $f \in C[a,b]$ and $g_k = f - (f(a)+1)\varphi_k$ also belongs to $C[a,b]$ and $g_k(a) = -1 < 0$, but

$$\|f - g_k\|_1 = \int_a^b |f(x) - g_k(x)|dx = \frac{f(a)+1}{2k} \to 0$$

as $k \to \infty$.

9. Show that $[a, b]$ can be expressed as the intersection of countable open intervals. It shows in particular that countable intersection of open sets may not be open.

   Solution. Simply observe

$$[a, b] = \bigcap_{j=1}^{\infty} (a - 1/j, b + 1/j) \ .$$

10. Optional. Show that every open set in $\mathbb{R}$ can be written as a countable union of disjoint open intervals. Suggestion: Introduce an equivalence relation $x \sim y$ if x and y belongs to the same open interval in the open set and observe that there are at most countable many such intervals.

    Solution.

    Let V be open in $\mathbb{R}$. Fix x $\in V$, there exists some open interval I, x $\in I$, $I \subseteq V$. Let $I_\alpha = (a_\alpha, b_\alpha)$, $\alpha \in \mathcal{A}$, be all intervals with this property. Let

$$I_x = (a_x, b_x), a_x = \inf_\alpha a_\alpha, b_x = \sup_\alpha b_\alpha.$$

   satisfy x $\in I_x$, $I_x \subseteq V$ (the largest open interval in V containing x). It is obvious that $I_x \cap I_y \neq \phi \Rightarrow I_x = I_y$. Let $x \sim y$ if $I_x = I_y$. Then one can show that $\sim$ is an equivalence relation. By the discussion above, we have

$$V = \bigcup_{x \in V} I_x = \bigcup_{[x] \in V/\sim} \left(\bigcup_{y \sim x} I_x \right) = \bigcup_{[x] \in V/\sim} I_x,$$

   which is a disjoint union. Moreover $V/\sim$ is at most countable since we can pick a rational number in each I_x to represent the class $[x] \in V/\sim$. Thus V can be written as a countable union of disjoint open intervals.

11. Fill in a proof of Proposition 2.8(b).

    Solution. $\Rightarrow$). Assume on the contrary there is an open G whose $f^{-1}(G)$ is not open, that is, there is some $x \in f^{-1}(G)$ so that the balls $B_{1/n}(x)$ always intersect the outside of $f^{-1}(G)$. Pick $x_n \in B_{1/n}(x)$ lying outside $f^{-1}(G)$. Since G is open, fix some $B_r(f(x)) \subset G$. That x_n lying outside $f^{-1}(G)$ implies $f(x_n)$ lying outside G, and in particular $B_r(f(x))$, so $\rho(f(x_n), f(x)) \geq r > 0$ for all n. On the other hand, as $x_n \to x$, by the continuity of f at x. We have, $f(x_n) \to f(x)$, that is, $\rho(f(x_n), f(x)) \to 0$, contradiction holds.

    $\Leftarrow$). Let $x_n \to x$ in X. The metric ball $B_\varepsilon(f(x))$ is open in Y, by assumption $f^{-1}(B_\varepsilon(f(x)))$ is an open set containing x. Hence we can find some metric ball $B_\delta(x) \subset f^{-1}(B_\varepsilon(f(x))$. As $x_n \to x$, there is some n_0 such that $x_n \in B_\delta(x)$ for all $n \geq n_0$. Hence, $f(x_n) \in B_\varepsilon(f(x))$ for all $n \geq n_0$, done.